An Inquiry-Based Project in a Modern Geometry Classroom

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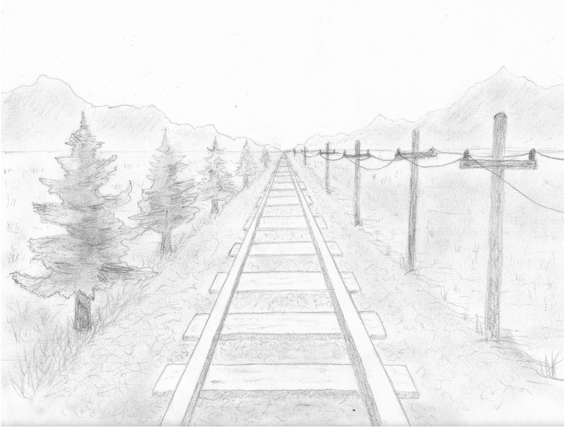
2024 MathFest

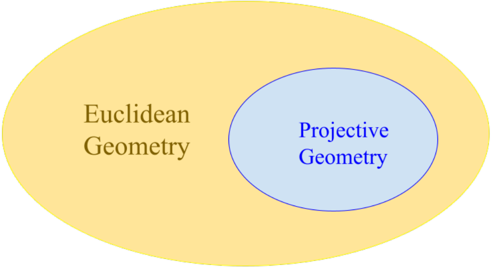
Outline

The classical approach to projective geometry



What geometric properties remain unchanged under projections?

The classical approach



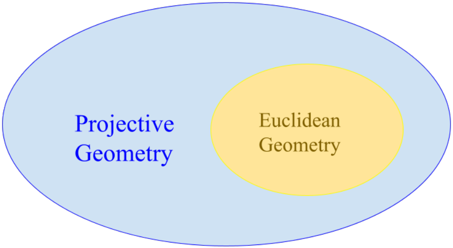
The Modern Approach

Definition (Euclidean geometry)

The study of invariant properties under Euclidean transformations.

Definition (Projective geometry)

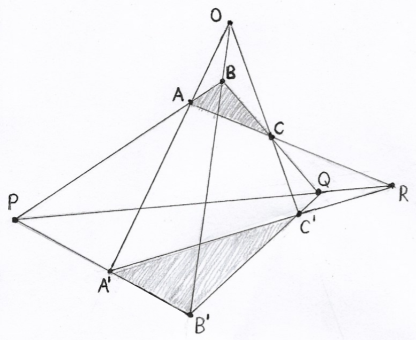
The study of invariant properties under projective transformations.



Desargues' Theorem in an Euclidean Plane

Theorem (Desargues')

If two triangles ABC and $A'B'C'$ are perspective from a point, then the three points of intersection of corresponding sides are collinear.

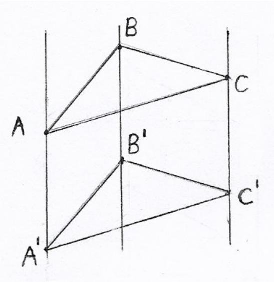


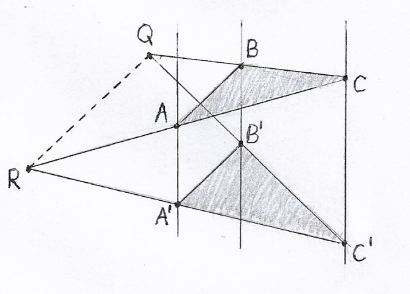
Corollary (1)

Let two triangles ABC and $A'B'C'$ be placed so that $AA' \parallel BB' \parallel CC'$. If $AB \parallel A'B'$, and $BC \parallel B'C'$, then $AC \parallel A'C'$.

Corollary (2)

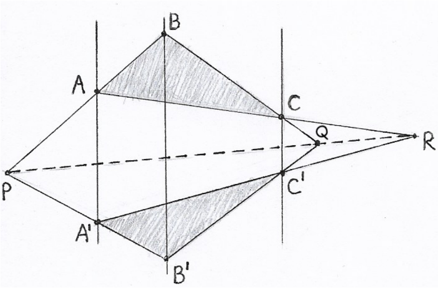
Let two triangles ABC and $A'B'C'$ be placed so that $AA' \parallel BB' \parallel CC'$. If $AB \parallel A'B'$, then $QR \parallel AB$, where $Q = BC \times B'C'$ and $R = AC \times A'C'$.





Corollary (3)

Let two triangles ABC and $A'B'C'$ be placed so that $AA' \parallel BB' \parallel CC'$, and no pairs of sides are parallel. Then P, Q, and R are collinear, where $P = AB \times A'B'$, $Q = BC \times B'C'$ and $R = AC \times A'C'$.

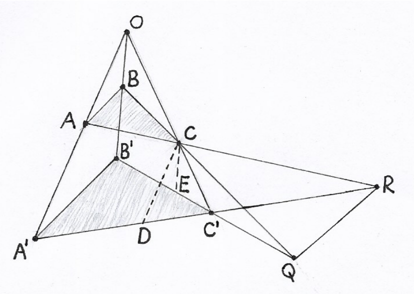


Corollary (4)

Let two triangles ABC and $A'B'C'$ be placed so that AA', BB', and CC' are concurrent at point O. If $AB \parallel A'B'$, and $BC \parallel B'C'$, then $AC \parallel A'C'$.

Corollary (5)

Let two triangles ABC and $A'B'C'$ be placed so that AA', BB', and CC' are concurrent at point O. If $AB \parallel A'B'$, then $QR \parallel AB$, where $Q = BC \times B'C'$ and $R = AC \times A'C'$.



Proof I.

Let D and E be points on $A'C'$ and $B'C'$ such that $CD \parallel OA'$ and $CE \parallel OB'$. Then

$$\left.\begin{array}{l} \text{in } \triangle RAA', CD \parallel OA' \Rightarrow \frac{RA}{RC} = \frac{AA'}{CD} \\ \text{in } \triangle C'OA', CD \parallel OA' \Rightarrow \frac{CC'}{OC'} = \frac{CD}{OA'} \end{array}\right\} \Rightarrow \frac{RA}{RC} \cdot \frac{CC'}{OC'} = \frac{AA'}{OA'}$$

$$\Rightarrow \frac{RA}{RC} = \frac{AA'}{OA'} \cdot \frac{OC'}{CC'}$$

$$\left.\begin{array}{l} \text{in } \triangle OB'C', CE \parallel OB' \Rightarrow \frac{CC'}{OC'} = \frac{CE}{OB'} \\ \text{in } \triangle QBB', CE \parallel OB' \Rightarrow \frac{QB}{QC} = \frac{BB'}{CE} \end{array}\right\} \Rightarrow \frac{CC'}{OC'} \cdot \frac{QB}{QC} = \frac{BB'}{OB'}$$

$$\Rightarrow \frac{QB}{QC} = \frac{BB'}{OB'} \cdot \frac{OC'}{CC'}$$

The assumption of $AB \parallel A'B'$ implies $\frac{AA'}{OA'} = \frac{BB'}{OB'}$, along with the above two obtained equalities, we have $\frac{RA}{RC} = \frac{QB}{QC}$, which implies $QR \parallel AB$. $\square$

Proof II.

It suffices to show that $\frac{QB}{QC} = \frac{RA}{RC}$. We use Menelaus theorem to calculate this ratios.

By Menelaus theorem, $\frac{QB}{QC} \cdot \frac{C'C}{C'O} \cdot \frac{B'O}{B'B} = 1$, which is equivalent to $\frac{QB}{QC} = \frac{C'O}{C'C} \cdot \frac{B'B}{B'O}$.

Similarly, we obtain $\frac{RA}{RC} = \frac{C'O}{C'C} \cdot \frac{A'A}{A'O}$ by considering the three collinear points C', R, and A' on each of the three sides of $\triangle AOC$. Notice that $\frac{A'A}{A'O} = \frac{B'B}{B'O}$ since $AB \parallel A'B'$. Now the equalities 1 and 2 implies $\frac{QB}{QC} = \frac{RA}{RC}$. $\square$

A Problem Solving Strategy

Example

Let P be a point inside a parallelogram $ABCD$. Points M and N are the midpoints of PB and PC, respectively. Let $O = AC \times BD$, and $Q = AN \times DM$. Prove that the points P, Q, and O are on the same line;

Guided questions

1. Give a reason why Desargues Theorem is a projective theorem.
2. Can we interpret the Theorem in a Euclidean plane? For instance, what happens if point O is an ideal point? Restate the Theorem without using the term *ideal point*.
3. Restate the Theorem when all three points O, P, and Q are ideal points.
4. In what other cases can you interpret the Theorem?
5. You have obtained different corollaries of the Desargues Theorem. Can you prove them without knowing the original theorem?
6. Which do you think is a larger geometry, Euclidean or projective?
7. What important method of investigation in mathematics have you learned in the classical approach of projective geometry?
8. Throughout history, from the ancient Egyptians and Euclid to Poncelet, geometry has been based on the concept of measurement. What new way of defining geometry have you learned in the modern approach of projective geometry?

Outcomes from a seemingly simple inquiry

1. Five Corollaries and three original proofs
2. An important method of investigation
3. An insight
4. A Problem Solving Strategy
5. Ideas for more projects (For example, replace Desargues Thm by Pascal Thm)
6. Appreciation of an influential mathematical philosophy
7. Great mathematical ideas may within the reach of a curious student

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